MICROLOCAL LIFTS OF EIGENFUNCTIONS ON HYPERBOLIC SURFACES AND TRILINEAR INVARIANT FUNCTIONALS

ANDRE REZNIKOV

Abstract. In [Z1] S. Zelditch introduced an equivariant version of a pseudo-differential calculus on a hyperbolic Riemann surface. We recast his construction in terms of trilinear invariant functionals on irreducible unitary representations of $PGL_2(\mathbb{R})$. This allows us to use certain properties of these functionals in the study of the action of pseudo-differential operators on eigenfunctions of the Laplacian on hyperbolic Riemann surfaces.

1. Introduction

1.1. **Motivation.** Let Y be a compact Riemann surface with a Riemannian metric of constant curvature -1 and the associated volume element dv. The corresponding Laplace-Beltrami operator Δ is non-negative and has purely discrete spectrum on the space $L^2(Y, dv)$ of functions on Y. We will denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq ...$ the eigenvalues of Δ and by $\phi_i = \phi_{\mu_i}$ the corresponding eigenfunctions (normalized to have L^2 norm one). In the theory of automorphic functions the functions ϕ_{μ_i} are called non-holomorphic forms, Maass forms (after H. Maass, [M]) or simply automorphic functions.

The study of Maass forms is important in analysis and in other areas. It has been understood since the seminal works of A. Selberg [Se] and I. Gel'fand, S.Fomin [GF] that representation theory plays an important role in this study. Central for this role is the correspondence between eigenfunctions of Laplacian on Y and unitary irreducible representations of the group $PGL_2(\mathbb{R})$ (or what is more customary of $PSL_2(\mathbb{R})$). This correspondence allows one, quite often, to obtain results that are more refined than similar results for the general case of a Riemannian metric of variable curvature.

A framework where the correspondence between eigenfunctions and representations plays a decisive role is the equivariant pseudo-differential calculus constructed by S. Zelditch in [Z1]-[Z3]. His motivation was to give a proof of the celebrated quantum ergodicity theorem of A. Shnirelman for hyperbolic surfaces (see Shnirelman [Sh], Y. Colin de Verdière [CdV], Zelditch [Z2]). The main ingredient of the proof of quantum ergodicity is a construction for each eigenfunction ϕ_i on Y of an associated probability measure dm_i on the spherical bundle $S^*(Y)$ of the co-tangent bundle of Y. The idea to associate such measures to eigenfunctions was a deep insight of Shnirelman. The measures dm_i are called microlocal lifts or micro-localizations of the corresponding eigenfunctions ϕ_i. The main property of these measures is that they satisfy the Egorov-type theorem, that is, the measures

dm_i are asymptotically invariant under the geodesic flow as $\mu_i \to \infty$. The measures dm_i are constructed in two steps. First, one constructs the so-called Wigner distribution dU_i or the auto-correlation distribution corresponding to ϕ_i. Namely, the distribution $dU_i \in \mathcal{D}(S^*(Y))$ such that for any pseudo-differential operator (PDO) A of order 0 with the symbol $a \in C^\infty(S^*(Y))$ the relation $< A\phi_i, \phi_i >= \int_{S^*(Y)} a\, dU_i$ holds. The distribution dU_i depends on a choice of the pseudo-differential calculus on Y. Next, one modifies dU_i (which are not non-negative) in order to get a probability measure dm_i asymptotically close to dU_i as $\mu_i \to \infty$. Such a modification is not unique. The Quantum Ergodicity Theorem claims that the measures dm_i converge to the standard Liouville measure on $S^*(Y)$, at least along a sequence of full density.

1.2. **Results.** In this paper we discuss a relation of the measures dm_i from Zelditch's version of the equivariant pseudo-differential calculus on Y to representation theory of the group $PGL_2(\mathbb{R})$. Namely, we will show how the asymptotic invariance under the geodesic flow of these measures follows from the uniqueness of invariant trilinear functionals on three irreducible unitary representations of $PGL_2(\mathbb{R})$. This is based on the following theorem which is the main underlying observation of the paper and comes from the uniqueness of trilinear functionals and Zelditch's description of pseudo-differential operators. To state it we recall first some basic facts about $S^*(Y)$ and differential operators on this space (see the standard excellent source [G6]).

It is well-known (and fundamental for our approach) that there is a transitive action of the group $G = PGL_2(\mathbb{R})$ on $S^*(Y)$(usually one considers the action of the group $PSL_2(\mathbb{R})$, but for some technical reasons explained in Section 4.2, we prefer to work with $PGL_2(\mathbb{R})$). Let $\mathcal{C}$ be the Casimir operator acting on the space $C^\infty(S^*(Y))$ of smooth functions on $S^*(Y)$ (this is the unique up to a constant second order hyperbolic G-invariant differential operator). The set of eigenvalues of $\mathcal{C}$ coincides with the set of eigenvalues of Δ on Y (although eigenspaces of $\mathcal{C}$ are infinite dimensional). Let V_μ be a μ-eigenspace of $\mathcal{C}$ which is *irreducible* under the G-action. For an eigenvalue μ the μ-eigenspace splits into a direct sum of finitely many irreducible ones and their span over all μ_i is dense in $C^\infty(S^*(Y))$ (see 3.2). It turns out that the space V_μ is a unitarizable irreducible representation of G. The representation V_μ is called an automorphic representation. All unitarizable irreducible representation of G are classified (see [G5] and Section 3 below). It turns out that any irreducible unitarizable representation has a dense subspace (called a space of smooth vectors) which could be realized as a quotient of the space $C^\infty(S^1)$ by a finite-dimensional subspace. Hence for any space V_μ there exists a map $\nu_\mu : C^\infty(S^1) \to V_\mu \subset C^\infty(S^*(Y))$. We assume for simplicity that ν_μ has no kernel and that $\mu \geq \frac{1}{4}$. It turns out that in this case the map ν_μ gives rise to an isometry $L^2(S^1) \to L^2(S^*(Y))$ and we denote by $<,>_{S^1}$, $<,>_Y$ the corresponding scalar products and the corresponding pairing between distributions and functions. The representation V_μ with the above property is called a class one representation of G of principal series; for V_μ which is not of class one the map ν_μ has a finite dimensional non-zero kernel. Such a representation V_μ is called a discrete

series representation. We deal with these also (see 5.3). Hence, any eigenfunction (in V_μ) of $\mathcal{C}$ is of the form $\nu_\mu(v)$ for an appropriate function $v \in C^\infty(S^1)$.

Let Op be the pseudo-differential calculus of Zelditch (we recall briefly the construction of Op in Section 2). In particular, this calculus assigns to a symbol $a(x,\lambda) \in C^\infty(S^*(Y)\times\mathbb{R}^+)$ an operator $Op(a)$ acting on $C^\infty(Y)$.

We will be interested in pseudo-differential operators of order 0 and moreover in those with the symbol independent of λ (see 2.3). This means that we consider the correspondence between symbols $a \in C^\infty(S^*(Y))$ and operators $Op(a)$ acting on $C^\infty(Y)$. For Maass forms Zelditch found a description of the action of such pseudo-differential operators in terms of the Helgason transform (which is a non-Euclidian analog of the Fourier transform). We rephrase his description in terms of representation theory as follows. Let μ be an eigenvalue of Δ and $E_\mu \subset C^\infty(Y)$ the corresponding eigenspace and let W_μ be the corresponding eigenspace of $\mathcal{C}$ (we have $W_\mu \simeq E_\mu \otimes V_\mu$). It turns out that one can construct a map $\mathcal{M} : E_\mu \to W_\mu^* \subset \mathcal{D}(S^*(Y))$ (called microlocalization) from the space of Maass forms to the space of distributions on $S^*(Y)$ such that for any symbol a and any Maass form $\phi \in E_\mu$ we have $Op(a)\phi(z) = \int_{S^1} a(zt)\mathcal{M}(\phi)(zt)dt$, where $z \in Y$ and the integration is along the fiber of $S^*(Y) \to Y$ (i.e. $Op(a)\phi$ is the push-forward of the distribution $a\mathcal{M}(\phi)$ to Y. It turns out that the result is a smooth function on Y and hence the integration is well-defined). Using this interpretation, we can express the action of a pseudo-differential operator on an eigenfunction as the multiplication of the corresponding distribution by the (smooth) symbol of the operator. This allows us to relate pseudo-differential operators to multiplication of automorphic functions and then to trilinear invariant functionals on representations.

We state now our main theorem (see 5.3)

Theorem. *Let $V_\mu \subset C^\infty(S^*(Y))$ be an irreducible eigenspace and $\nu_\mu : C^\infty(S^1) \to V_\mu$ the corresponding map and let μ_1, μ_2 be eigenvalues of the Laplacian Δ. There exists an explicit distribution $l_{\alpha,\beta,\gamma} \in \mathcal{D}(S^1)$ on S^1 depending on three complex parameters $\alpha,\beta,\gamma \in \mathbb{C}$ such that for any symbol a of the form $a = \nu_\mu(v_a) \in C^\infty(S^*(Y))$ with $v_a \in C^\infty(S^1)$ and ϕ_1, ϕ_2 eigenfunctions of the Laplacian Δ, $\Delta\phi_i = \mu_i\phi_i$, there exists a constant $a_{\mu,\mu_1,\mu_2} \in \mathbb{C}$ satisfying the relation*

$$< Op(a)\phi_1, \phi_2 >_Y = a_{\mu,\mu_1,\mu_2} \cdot < l_{\mu,\mu_1,\mu_2}, v_a >_{S^1} \quad . \tag{1}$$

Hence for the special kind of symbols, which we call irreducible symbols (i.e. those belonging to one of the irreducible representations V_μ) we are able to analyze the action of the corresponding pseudo-differential operator on eigenfunctions by means of representation theory. We note that the space spanned by such symbols is dense in the space of all smooth symbols.

We want to stress that for us the most important conclusion of the theorem above is the claim that the distribution l_{μ,μ_1,μ_2} has an *explicit* kernel which depends only on eigenvalues

and not on the choice of eigenfunctions ϕ_i nor on the choice of the *symbol* a. We will use this heavily throughout the paper. We will see that one can choose the kernel of $l_{\alpha,\beta,\gamma}$ to be given by a function on S^1 which is similar to the function $|\sin(\theta)|^{\frac{-1-\lambda}{2}}$ with λ pure imaginary (see (37)).

The coefficients a_{μ,μ_i,μ_j} depend on the choice of ϕ_i and ϕ_j and encode an important information about corresponding eigenfunctions (for arithmetic surfaces and a special basis of eigenfunctions, the Hecke-Maass basis, these coefficients are connected to special values of certain L-functions, see [Sa]). We will discuss bounds on these coefficients as functions of eigenvalues μ_i and make some far reaching conjectures about their size (see 4.4).

In Section 4 we show how to express the setting of pseudo-differential operators in terms of trilinear invariant functionals on irreducible representations of G. The main technical fact about trilinear invariant functionals we use in this paper, beside their uniqueness, is that such functionals could be described in terms of an explicit kernel. We study this kernel from the point of view of oscillatory integrals. Once we relate the distribution l_{μ,μ_i,μ_j} to a trilinear functional, we are able to give an explanation for the asymptotic invariance of the microlocal measures in terms of the geometry of the phase of this kernel. We also explain why some probability measures suggested by the construction of S. Wolpert ([Wo]) are asymptotic corrections to distributions dU_i. This gives the positivity result necessary in the Shnirelman's argument.

We note that invariant trilinear functionals play an important role in [Z3], albeit implicitly. Essentially, different iterative formulas in [Z3] (which were developed in order to prove the asymptotic invariance in the first place) follow from the uniqueness of invariant trilinear functionals (we note that these formulas served as a starting point for the recent approach of E. Lindenstrauss to the quantum unique ergodicity, see [Li]). The approach taken in [Z3] is based on differential relations coming from the action of the Lie algebra $sl_2(\mathbb{R})$ while our approach is based on properties of integral operators involved and hence, in principle, is more flexible. While the uniqueness of invariant trilinear functionals is widely known to specialists in automorphic functions (where it plays an important role in the theory of L-functions) it is rarely used by analysts and deserves a wider recognition (see the recent book [U] however).

The paper is organized as follows. We begin with a brief review of Zelditch's construction of the equivariant pseudo-differential calculus (see [Z1] for more detail) and the well-known relation between eigenfunctions on Y and representation theory of $PGL_2(\mathbb{R})$ (due to Gel'fand and Fomin, see [GF], [G6]). We also express Zelditch's distributions dU_i in terms of special vectors (distributions) in the corresponding automorphic representations. We then introduce our main tool of invariant trilinear functionals and recast pseudo-differential calculus in these terms. We next describe invariant trilinear functionals explicitly in terms of their kernels. Central for this is the alluded above uniqueness of invariant trilinear functionals. It turns out that one can choose such kernels to be given by simple homogeneous functions on copies of $\mathbb{R}^2 \setminus 0$. To see this we use the standard

model of the irreducible unitary representations of $PGL_2(\mathbb{R})$ realized in homogeneous functions on $\mathbb{R}^2 \setminus 0$. We use this explicit description of trilinear functionals in order to deduce the asymptotic invariance of microlocal lifts of eigenfunctions (Theorem 6.2) and to construct asymptotic probability measures (Theorem 7.1). Both results follow from the explicit form of the kernel of trilinear functional and the stationary phase method. We also give a quantitative bound on the non-invariant part in terms of an appropriate Sobolev norm. On the basis of our analysis we show that one might expect that matrix coefficients $< A\phi_i, \phi_i >$ are invariant under the geodesic flow up to a higher order (by the factor $\mu_i^{-\frac{1}{4}}$) than the Egorov's theorem predicts (this was also noticed by Zelditch). We also show that for a fixed pseudo-differential operator A the spectral density of $A\phi_i$ is (essentially) supported in a short interval near μ_i (Theorem 8.1) and formulate a conjecture concerning the size of coefficients $< A\phi_i, \phi_j >$ on this interval.

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2. Equivariant pseudo-differential operators

We describe the construction of Zelditch [Z1] of the equivariant pseudo-differential calculus on a hyperbolic surface. It is based on Helgason's representation theorem for eigenfunctions on the unit disk D.

2.1. **Geometric setting.** We begin with some well-known definitions ([He],[Z1]). Let D be the Poincaré open unit disk with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/(1 - r^2)^2$ and the hyperbolic volume element $dvol_H = dxdy/(1 - r^2)^2$, where $r^2 = x^2 + y^2$. We denote by B the boundary circle of D (on infinity) . Given a pair $(z, b) \in D \times B$ let $\xi(z, b)$ be the unique horocycle through $z \in D$ with forward end point $b \in B$. The non-Euclidian (signed) distance from the origin 0 to $\xi(z, b)$ will be denoted $< z, b >$. It is well known that functions $e^{(\frac{\lambda-1}{2})<z,b>}$ are eigenfunctions of the hyperbolic Laplacian with the eigenvalue $\mu = \frac{1-\lambda^2}{4}$ (here we slightly changed normalization from the one adopted in [Z1]). The group $PSU(1, 1) \simeq PSL_2(\mathbb{R})$ acts by the standard fractional linear transformations on D and coincides with group of isometries of D. We will use the identification $D \times B \simeq PSL_2(\mathbb{R})$ via the equivariant map sending a pair (z, b) to the unique element $g_{z,b} \in PSL_2(\mathbb{R})$ such that $g_{z,b} \cdot 0 = z$ and $g_{z,b} \cdot 1 = b$. One can view this as a well-known identification $PSL_2(\mathbb{R}) \simeq S^*(D) \simeq S(D)$ with the (co-)spherical bundle on D. The action of $g \cdot (z, b) \to (gz, gb)$ coincides then with the left action of $PSL_2(\mathbb{R})$ on itself.

We choose $\Gamma \subset PSL_2(\mathbb{R})$ a (co-compact) discrete subgroup such that the Riemann surface $Y = \Gamma \simeq D$.

2.2. **Helgason's representation.** In [He] Helgason proved the following

Theorem. *Let $\phi \in C^\infty(D)$ of at most polynomial growth (in the hyperbolic distance from the origin) near the boundary B and satisfying $\Delta\phi = \frac{1-\lambda^2}{4}\phi$. Then there exists a distribution on the boundary $T \in \mathcal{D}(B)$ such that*

$$\phi(z) = \int_B e^{(\frac{1+\lambda}{2})<z,b>} dT(b) \ . \tag{2}$$

We denote the correspondence defined by (2) by $\mathcal{P}_\lambda : \mathcal{D}(B) \to C^\infty(D)$ and refer to it as the Helgason map (it is also called the non-Euclidian Poisson map). An important point is that Helgason's representation is equivariant with respect to the standard action of $PSL_2(\mathbb{R})$ on D and the following twisted action on B. Namely, let π_λ be the representation of $SL_2(\mathbb{R})$ on the space of functions (or distributions) defined by

$$\pi_\lambda(g)f(b) = f(g^{-1} \cdot b)|g'(b)|^{\frac{\lambda-1}{2}} \ . \tag{3}$$

This defines a representation of $PSL_2(\mathbb{R})$ (which is unitary and irreducible in the space $L^2(B)$ for $\lambda \in i\mathbb{R}$). We have then $\mathcal{P}_\lambda(\pi_\lambda(g)T)(z) = \mathcal{P}_\lambda(T)(g^{-1}z)$ for any $g \in PSL_2(\mathbb{R})$. In particular, an eigenfunction ϕ is Γ-invariant if and only if the distribution T is (π_λ, Γ)-invariant (we will see latter that this is exactly the Frobenius reciprocity from the theory of automorphic functions; see 3.4).

We note that there is an inverse to $\mathcal{P}$ map given by (properly defined) boundary values of eigenfunctions (see [He],[Le]).

Similar to (2) one have the following Helgason non-Euclidian Fourier transform $\mathcal{F} : C_0^\infty(D) \to C^\infty(\mathbb{R}^+ \times B)$ for a general function $f \in C_0^\infty(D)$:

$$\mathcal{F}(f)(\lambda, b) = \hat{f}(\lambda, b) = \int_D e^{(\frac{1-\lambda}{2})<z,b>} f(z) dvol(z) \tag{4}$$

and the inverse transform

$$f(z) = \hat{f}(\lambda, b) = \int_{\mathbb{R}^+ \times B} e^{(\frac{1+\lambda}{2})<z,b>} \hat{f}(\lambda, b) \lambda \tanh(\pi\lambda/2) d\lambda db \ . \tag{5}$$

The non-Euclidian Fourier transform $\mathcal{F}$ is an isometry between spaces $L^2(D, dvol_H)$ and $L^2(\mathbb{R}^+ \times B, (1/2\pi)\tanh(\pi\lambda/2)d\lambda db)$.

2.3. **Pseudo-differential operators.** Based on the representation (5) Zelditch introduced in [Z1] the following form of $SL_2(\mathbb{R})$-equivariant pseudo-differential calculus.

Given any operator $A : C^\infty(D) \to C^\infty(D)$ one defines its complete symbol $a(z, \lambda, b) \in C^\infty(D \times \mathbb{R}^+ \times B)$ by

$$A e^{(\frac{1+\lambda}{2})<z,b>} = a(z, \lambda, b) e^{(\frac{1+\lambda}{2})<z,b>} \ . \tag{6}$$

By the inversion formula (5), we have the following representation

$$Af(z) = 1/2\pi \int_{\mathbb{R}^+\times B} e^{(\frac{1+\lambda}{2})<z,b>} a(z,\lambda,b)\hat{f}(\lambda,b)\lambda \tanh(\pi\lambda/2) d\lambda db \ . \tag{7}$$

It is assumed that the symbol of A has the standard asymptotic (in the symbol topology) expansion $a \sim \sum_0^\infty \lambda^{-j} a_{-j}(z,b)$ as $|\lambda| \to \infty$. We will be interested in pseudo-differential operators of order 0 and hence will assume that the symbol is independent of λ. For such a symbol $a(z,b) \in C^\infty(D\times B)$ we will denote $Op(a)$ the pseudo-differential operator defined by (7).

The correspondence between operators A and their symbols $a(z,\lambda,b)$ is equivariant. Namely, the symbol of gA is given by $a(gz,\lambda,gb)$. We will be interested in Γ-invariant version of pseudo-differential operators, i.e. those which commute with the action of Γ. Such symbols naturally gives rise to the pseudo-differential operators on the Riemann surface Y. Let $\phi \in C^\infty(\Gamma\setminus D) \simeq C^\infty(Y)$ be an eigenfunction of Δ with the eigenvalue $\mu = \frac{1-\lambda^2}{4}$ and $T \in \mathcal{D}(B)$ be the boundary distribution assigned to ϕ via Helgason's representation (2). Zelditch then showed in [Z1] that for any Γ-invariant symbol $a(z,b) \in C^\infty(D\times B)^\Gamma$ we have as above

$$Op(a)\phi(z) = \int_B a(z,b) e^{(\frac{1+\lambda}{2})<z,b>} dT(b) \ . \tag{8}$$

This formula will serve us as a starting point for an interpretation of $Op(a)$ in terms of representation theory and particularly in terms of trilinear invariant functionals.

3. Representation theory and eigenfunctions

We recall the standard connection between eigenfunctions and representation theory (see [G6]).

3.1. **Automorphic representations.** Let us describe the geometric construction which allows one to pass from analysis on a Riemann surface to representation theory.

One stars with the Poincaré unit disk D as above (or equivalently $\mathbb{H}$ the upper half plane with the hyperbolic metric of constant curvature -1; the use of $\mathbb{H}$ is more customary in the theory of automorphic functions). The group $SL_2(\mathbb{R}) \simeq SU(1,1)$ acts on D (or $\mathbb{H}$) by fractional linear transformations. This action allows to identify the group $PSL_2(\mathbb{R})$ with the group of all orientation preserving motions of D. For reasons explained bellow we would like to work with the group G of all motions of D; this group is isomorphic to $PGL_2(\mathbb{R})$. Hence throughout the paper we denote $G = PGL_2(\mathbb{R})$.

Let us fix a discrete co-compact subgroup $\Gamma \subset G$ and set $Y = \Gamma\setminus D$. We consider the Laplace operator on the Riemann surface Y and denote by μ_i its eigenvalues and by ϕ_i the corresponding normalized eigenfunctions.

The case when Γ acts freely on D precisely corresponds to the case discussed in 1.1 (this follows from the uniformization theorem for the Riemann surface Y). Our results hold

for general co-compact subgroup Γ (and in fact, with slight modifications, for any lattice $\Gamma \subset G$).

We will identify the upper half plane $\mathbb{H}$ (or D) with G/K, where $K = PO(2)$ is a maximal compact subgroup of G (this follows from the fact that G acts transitively on $\mathbb{H}$ and the stabilizer in G of the point $z_0 = i \in \mathbb{H}$ coincides with K).

We denote by X the compact quotient $\Gamma \setminus G$ (we call it the automorphic space). In the case when Γ acts freely on $\mathbb{H}$ one can identify the space X with the bundle $S(Y)$ of unit tangent vectors to the Riemann surface $Y = \Gamma \setminus \mathbb{H}$.

The group G acts on X (from the right) and hence on the space of functions on X. We fix the unique G-invariant measure μ_X on X of total mass one. Let $L^2(X) = L^2(X, d\mu_X)$ be the space of square integrable functions and $(\Pi_X, G, L^2(X))$ the corresponding unitary representation. We will denote by P_X the Hermitian form on $L^2(X)$ given by the scalar product. We denote by $|| \ ||_X$ or simply $|| \ ||$ the corresponding norm and by $\langle f, g\rangle_X$ the corresponding scalar product.

The identification $Y = \Gamma \setminus \mathbb{H} \simeq X/K$ induces the embedding $L^2(Y) \subset L^2(X)$. We will always identify the space $L^2(Y)$ with the subspace of K-invariant functions in $L^2(X)$.

Let ϕ be a normalized eigenfunction of the Laplace-Beltrami operator on Y. Consider the closed G-invariant subspace $L_\phi \subset L^2(X)$ generated by ϕ under the action of G. It is well-known that $(\pi, L) = (\pi_\phi, L_\phi)$ is an irreducible unitary representation of G (see [G6]).

Usually it is more convenient to work with the space $V = L^\infty$ of smooth vectors in L. The unitary Hermitian form P_X on V is G-invariant.

A smooth representation (π, G, V) equipped with a positive G-invariant Hermitian form P we will call a *smooth pre-unitary representation*; this simply means that V is the space of smooth vectors in the unitary representation obtained from V by completion with respect to P.

Thus starting with an automorphic function ϕ we constructed an irreducible smooth pre-unitary representation (π, V). In fact we constructed this space together with a canonical morphism $\nu : V \to C^\infty(X)$ since $C^\infty(X)$ is the smooth part of $L^2(X)$.

Definition. A smooth pre-unitary representation (π, G, V) equipped with a G-morphism $\nu : V \to C^\infty(X)$ we will call an X-*enhanced representation*.

We will assume that the morphism ν is normalized, i.e. it carries the standard L^2 Hermitian form P_X on $C^\infty(X)$ into Hermitian form P on V.

Thus starting with an automorphic function ϕ we constructed

(i) An X-enhanced irreducible pre-unitary representation (π, V, ν),

(ii) A K-invariant unit vector $e_V \in V$ (this vector is just our function ϕ).

Conversely, suppose we are given an irreducible smooth pre-unitary X-enhanced representation (π, V, ν) of the group G and a K-fixed unit vector $e_V \in V$. Then the function $\phi = \nu(e_V) \in C^\infty(X)$ is K-invariant and hence can be considered as a function on Y. The fact that the representation (π, V) is irreducible implies that ϕ is an automorphic function, i.e. an eigenfunction of Laplacian on Y.

Thus we have established a natural correspondence between Maass forms ϕ and tuples (π, V, ν, e_V), where (π, V, ν) is an X-enhanced irreducible smooth pre-unitary representation and $e_V \in V$ is a unit K-invariant vector.

3.2. **Decomposition of the representation** $(\Pi_X, G, L^2(X))$**.** It is well known that for X compact the representation $(\Pi_X, G, L^2(X))$ decomposes into a direct (infinite) sum

$$L^2(X) = \oplus_j (\pi_j, L_j) \tag{9}$$

of irreducible unitary representations of G (all representations appear with finite multiplicities (see [G6])). Let (π, L) be one of these irreducible "automorphic" representations and $V = L^\infty$ its smooth part. By definition V is given with a G-equivariant isometric morphism $\nu : V \to C^\infty(X)$, i.e. V is an X-enhanced representation.

If V has a K-invariant vector it corresponds to a Maass form. There are other spaces in this decomposition which correspond to discrete series representations. Since they are not related to Maass forms we will not study them in more detail.

3.3. **Representations of** $PGL_2(\mathbb{R})$**.** All irreducible unitary representations of G are classified. For simplicity we consider first those with a nonzero K-fixed vector (so called representations of class one) since only these representations arise from Maass forms. These are the representations of the principal and the complementary series and the trivial representation.

We will use the following standard explicit model for irreducible smooth representations of G.

For every complex number λ consider the space V_λ of smooth even homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $\lambda - 1$ (which means that $f(ax, ay) = |a|^{\lambda-1} f(x, y)$ for all $a \in \mathbb{R} \setminus 0$). The representation (π_λ, V_λ) is induced by the action of the group $GL_2(\mathbb{R})$ given by $\pi_\lambda(g) f(x, y) = f(g^{-1}(x, y)) |\det g|^{(\lambda-1)/2}$. This action is trivial on the center of $GL_2(\mathbb{R})$ and hence defines a representation of G. The representation (π_λ, V_λ) is called *representation of the generalized principal series.*

When $\lambda = it$ is purely imaginary the representation (π_λ, V_λ) is pre-unitary; the G-invariant scalar product in V_λ is given by $\langle f, g\rangle_{\pi_\lambda} = \frac{1}{2\pi}\int_{S^1} f\bar{g}d\theta$. These representations are called representations of *the principal series.*

When $\lambda \in (-1, 1)$ the representation (π_λ, V_λ) is called a representation of the complementary series. These representations are also pre-unitary, but the formula for the scalar product is more complicated (see [G5]).

All these representations have K-invariant vectors. We fix a K-invariant unit vector $e_\lambda \in V_\lambda$ to be a function which is one on the unit circle in $\mathbb{R}^2$.

Representations of the principal and the complimentary series exhaust all nontrivial irreducible pre-unitary representations of G of class one. The rest of unitary irreducible representations of G could be realized as submodules (or quotients) in the spaces V_λ for $\lambda \in \mathbb{Z}$. These are called discrete series representations ([G5], [L]).

In what follows we will do necessary computations for representation of the principal series. Computations for the complementary series are a little more involved but essentially the same (compare with [BR1], section 5.5, where similar computations are described in detail).

Suppose we are given a class one X-enhanced representation $\nu : V_\lambda \to C^\infty(X)$; we assume ν to be an isometric embedding. Such ν gives rise to an eigenfunction of the Laplacian on the Riemann surface $Y = X/K$ as before. Namely, if $e_\lambda \in V_\lambda$ is a unit K-fixed vector then the function $\phi = \nu(e_\lambda)$ is a normalized eigenfunction of the Laplacian on the space $Y = X/K$ with the eigenvalue $\mu = \frac{1-\lambda^2}{4}$. This explains why λ is a natural parameter to describe Maass forms. We note that the Casimir operator $\mathcal{C}$ is a scalar operator on V_λ with the same eigenvalue. However, eigenspaces of $\mathcal{C}$ in $C^\infty(X)$ correspond only to isotypic components because of possible multiplicities.

3.4. **Helgason's representation and Frobenius reciprocity.** Here we reformulate Helgason's representation (2) for Γ-invariant eigenfunctions in terms of Frobenius reciprocity of Gel'fand and Fomin.

Let (π, G, V) be an irreducible unitary X-enhanced representation. We have the following **Frobenius reciprocity** ([G6], [Ol], [BR2]):

Theorem.

$$\mathrm{Mor}_G(V, C^\infty(X)) \simeq \mathrm{Mor}_\Gamma(V, \mathbb{C}) \ . \tag{10}$$

Namely, to every G-morphism $\nu : V \to C^\infty(\Gamma \setminus G)$ corresponds a Γ-invariant functional I on the space V given by $I(v) = \nu(v)(e)$ (here e is the identity in G). Given I we can recover ν as $\nu(v)(g) = I(\pi(g)v)$.

In particular, let π be of class one and $e_0 \in V$ be a unit K-fixed vector then the corresponding eigenfunction (i.e. the Maass form) on D (or $\mathbb{H}$) is given by

$$\phi(z) = I(\pi(g)e_0) \ , \tag{11}$$

with $g \cdot 0 = z \in D$ (or correspondingly $g \cdot i = z \in \mathbb{H}$).

This is exactly the Helgason's representation (2) if we view the automorphic functional I as a distribution on the space $V \simeq C^\infty(S^1)$.

Hence, Helgason's representation shows how to realize the K-fixed vector (i.e. the Maass form) on D. However, it does not show how to realize other vectors in V (and apart from

e_0 those could not be realized in the space of functions on D). Zelditch [Z2] noticed how to re-write Helgason's representation in a form appropriate for a general vector $v \in V$.

Namely, let us choose an identification $V \simeq V_\lambda$ and consider the following left Γ-invariant distribution on $D \times B$:

$$e_\pi(g) = e^{(\frac{1+\lambda}{2})<z,b>} d\mathrm{vol}(z) dT(b) \tag{12}$$

with $g = (z, b)$ under the identification $D \times B \simeq G$ in 2.1. We have $e_\pi \in \mathcal{D}(\Gamma \setminus G)$. It is easy to see that in terms of Frobenius reciprocity this distribution is nothing else than

$$e_\pi(g) = I(\pi(g)\delta), \tag{13}$$

where $\delta = \delta_1 = \sum_k e_{2k}$ is the distribution which is formally the sum of all K-types in the standard basis of V (see [Z2],[L]) or simply is equal to the Dirac delta distribution at $1 \in S^1$ in the realization $V \simeq V_\lambda \simeq C^\infty_{even}(S^1)$. We note (see [L], [Z2]) that unit vectors e_{2k} become exponents $e_{2k} = \exp(2\pi i 2k\theta)$ in the realization $V \simeq C^\infty_{even}(S^1)$ of the principal series representations of $PGL_2(\mathbb{R})$.

Hence, we see that the distribution e_π vanishes on functions which are orthogonal to $V \subset C^\infty(X)$ and on V takes value 1 on vectors in the standard basis $\{e_{2k}\}$. This is exactly the description given in [Z2] (Proposition 2.2). We will use the representation (13) extensively in what follows.

The distribution e_π gives rise to the imbedding $C^\infty_{even}(S^1) \to V \subset C^\infty(X)$, $v \mapsto \phi_v(g)$ via

$$\phi_v(g) = \int_K e_\pi(gk)\bar{v}(k \cdot 1) dk = I(\pi(g)v) \tag{14}$$

which again the isomorphism (10).

4. Trilinear invariant functionals

We introduce now the invariant trilinear functionals on irreducible representations which will be our main tool in what follows.

4.1. **Automorphic triple products.**

Suppose we are given three X-enhanced representations of G

$$\nu_j : V_j \to C^\infty(X), \quad j = 1, 2, 3 \ .$$

We define the G-invariant trilinear form $l^{aut}_{\pi_1,\pi_2,\pi_3} : V_1 \otimes V_2 \otimes V_3 \to \mathbb{C}$, by formula

$$l^{aut}_{\pi_1,\pi_2,\pi_3}(v_1 \otimes v_2 \otimes v_3) = \int_X \phi_{v_1}(x)\phi_{v_2}(x)\phi_{v_3}(x) d\mu_X \ , \tag{15}$$

where $\phi_{v_j} = \nu_j(v_j) \in C^\infty(X)$ for $v_j \in V_j$.

4.2. **Uniqueness of triple products.** The central fact about invariant trilinear functionals is the following uniqueness result:

Theorem. *Let* (π_j, V_j), $j = 1,2,3$, *be three irreducible smooth admissible representations of* G. *Then* $\dim \mathrm{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C}) \leq 1$.

Remark. The uniqueness statement was proven by Oksak in [O] for the group $SL(2, \mathbb{C})$ and the proof could be adopted for $PGL_2(\mathbb{R})$ as well (see also [Mo] and [Lo] for different proofs). For the p-adic $GL(2)$ more refined results were obtained by Prasad (see [P]). He also proved the uniqueness when at least one representation is a discrete series representation of $GL_2(\mathbb{R})$.

There is no uniqueness of trilinear functionals for representations of $SL_2(\mathbb{R})$ (the space is two-dimensional). This is the reason why we prefer to work with $PGL_2(\mathbb{R})$ (although the method could be easily adopted to $SL_2(\mathbb{R})$).

For $SL_2(\mathbb{R})$ one has the following uniqueness statement instead. Let (π, V) and (σ, W) be two irreducible smooth pre-unitary representations of $SL_2(\mathbb{R})$ of class one. Then the space of $SL_2(\mathbb{R})$-invariant trilinear functionals on $V \otimes V \otimes W$ which are symmetric in the first two variables is one-dimensional. This is the correct uniqueness result needed if one wants to work with $SL_2(\mathbb{R})$.

4.3. **Model trilinear functionals.** For every $\lambda \in \mathbb{C}$ we denote by (π_λ, V_λ) the smooth class one representation of the generalized principle series of the group $G = PGL_2(\mathbb{R})$ described in 3.3. We will use the realization of (π_λ, V_λ) in the space of smooth homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $\lambda - 1$.

For explicit computations it is often convenient to pass from plane model to a circle model. Namely, the restriction of functions in V_λ to the unit circle $S^1 \subset \mathbb{R}^2$ defines an isomorphism of the space V_λ with the space $C^\infty(S^1)^{even}$ of even smooth functions on S^1 so we can think about vectors in V_λ as functions on S^1.

We describe now the *model* invariant trilinear functional using the explicit geometric models for irreducible representations described above. Namely, for given three complex numbers λ_j, $j = 1,2,3$, we construct explicitly nontrivial trilinear functional l^{mod} : $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \to \mathbb{C}$ by means of its kernel.

4.3.1. *Kernel of* l^{mod}. Let $\omega(\xi, \eta) = \xi_1\eta_2 - \xi_2\eta_1$ be $SL_2(\mathbb{R})$-invariant of a pair of vectors $\xi,\ \eta \in \mathbb{R}^2$. We set

$$K_{\lambda_1,\lambda_2,\lambda_3}(s_1, s_2, s_3) = |\omega(s_2, s_3)|^{(\alpha-1)/2}\ |\omega(s_1, s_3)|^{(\beta-1)/2}|\omega(s_1, s_2)|^{(\gamma-1)/2} \tag{16}$$

for $s_1, s_2, s_3 \in \mathbb{R}^2 \setminus 0$, where $\alpha = \lambda_1 - \lambda_2 - \lambda_3, \beta = -\lambda_1 + \lambda_2 - \lambda_3, \gamma = -\lambda_1 - \lambda_2 + \lambda_3$.

The kernel function $K_{\lambda_1,\lambda_2,\lambda_3}(s_1, s_2, s_3)$ satisfies two main properties:

(1) K is invariant with respect to the diagonal action of $SL_2(\mathbb{R})$.

(2) K is homogeneous of degree $-1-\lambda_j$ in each variable s_j.

Hence if f_j are homogeneous functions of degree $-1+\lambda_j$, then the function

$$F(s_1,s_2,s_3)=f_1(s_1)f_2(s_2)f_3(s_3)K_{\lambda_1,\lambda_2,\lambda_3}(s_1,s_2,s_3)\ ,$$

is homogeneous of degree -2 in each variable $s_j\in\mathbb{R}^2\setminus 0$.

4.3.2. *Functional l^{mod}.* To define the model trilinear functional l^{mod} we notice that on the space $\mathcal{V}$ of functions of homogeneous degree -2 on $\mathbb{R}^2\setminus 0$ there exists a natural $SL_2(\mathbb{R})$-invariant functional $\mathfrak{L}:\mathcal{V}\to\mathbb{C}$. It is given by the formula $\mathfrak{L}(f)=\int_\Sigma f d\sigma$ where the integral is taken over any closed curve $\Sigma\subset\mathbb{R}^2\setminus 0$ which goes around 0 and the measure $d\sigma$ on Σ is given by the area element inside of Σ divided by π; this last normalization factor is chosen so that $\mathfrak{L}(Q^{-1})=1$ for the standard quadratic form Q on $\mathbb{R}^2$.

Applying $\mathfrak{L}$ separately to each variable $s_i\in\mathbb{R}^2\setminus 0$ of the function $F(s_1,s_2,s_3)$ above we obtain the G-invariant functional

$$l^{mod}_{\pi_1,\pi_2,\pi_3}(f_1\otimes f_2\otimes f_3):=\langle\mathfrak{L}\otimes\mathfrak{L}\otimes\mathfrak{L},F\rangle\ . \tag{17}$$

We call it the *model triple product* and denote by $l^{mod}_{\pi_1,\pi_2,\pi_3}$.

In the circle model this functional is expressed by the following integral:

$$l^{mod}_{\pi_1,\pi_2,\pi_3}(f_1\otimes f_2\otimes f_3)=(2\pi)^{-3}\iiint f_1(x)f_2(y)f_3(z)K_{\lambda_1,\lambda_2,\lambda_3}(x,y,z)dxdydz, \tag{18}$$

where $x,y,z\in S^1$ are the standard angular parameters on the circle and

$$K_{\lambda_1,\lambda_2,\lambda_3}(x,y,z)=|\sin(y-z)|^{(\alpha-1)/2}|\sin(x-z)|^{(\beta-1)/2}|\sin(x-y)|^{(\gamma-1)/2} \tag{19}$$

with $\alpha,\ \beta,\ \gamma\in i\mathbb{R}$ as before.

Remark. The integral defining the trilinear functional is often divergent and the functional should be defined using regularization of this integral. There are standard procedures how to make such a regularization (see e.g. [G1]).

4.4. **Coefficients of proportionality.** By the uniqueness principle, for automorphic representations π_1,π_2,π_3 there exists a constant a_{π_1,π_2,π_3} of proportionality between the *model* functional (17) and the *automorphic* functional (15) :

$$l^{aut}_{\pi_1,\pi_2,\pi_3}=a_{\pi_1,\pi_2,\pi_3}\cdot l^{mod}_{\pi_1,\pi_2,\pi_3}\ . \tag{20}$$

4.4.1. *Bounds on a_{π_1,π_2,π_3}.* In this paper we will need the following particular case of a general problem of estimating the coefficients a_{π_1,π_2,π_3}. Let us fix an automorphic representation $\pi_1\simeq\pi_\mu$ and let $\pi_2=\pi_3\simeq\pi_{\lambda_i}$ as $|\lambda_i|\to\infty$ through the set of parameters of automorphic representations of class one. We have the following (the so-called convexity) bound:

Proposition. *There exists an effective constant C such that for any π_μ and π_{λ_i}*

$$|a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}| \leq C(\max(|\mu|,|\lambda_i|))^{\frac{1}{2}} \ . \tag{21}$$

Proof. This follows from methods of [BR3]. For μ fixed and $|\lambda_i| \to \infty$ this is also shown in [Re] by a slightly different argument. We discuss similar bounds for the case of the representation π_μ of discrete series in the course of the proof of Theorem 6.2. □

4.4.2. *A conjecture.* The major problem in the theory of automorphic functions and analysis on Y is to find a method which would allow one to obtain better bounds for coefficients a_{π_1,π_2,π_3}.

We would like to make the following conjecture concerning the size of coefficients $a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}$:

Conjecture. *For a fixed π_μ and for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ independent of λ_i such that*

$$|a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}| \leq C_\varepsilon |\lambda_i|^\varepsilon \ ,$$

as $|\lambda_i| \to \infty$.

In a special case of a congruence subgroup Γ this conjecture is consistent with the Lindelöf conjecture from the theory of automorphic L-functions (see [Wa] for a connection to the theory of L-functions, [Sa] for the survey and [BR1], [BR3], [Re] for the connection to trilinear functionals).

5. PDO and microlocal distributions dU_i

In this section we piece together pseudo-differential operators and representation theory in order to express Zelditch's microlocal lifts of eigenfunctions in terms of representation theory.

5.1. **PDO.** Let $\phi \in C^\infty(Y)$ be an eigenfunction with the eigenvalue $\mu = \frac{1-\lambda^2}{4}$, (π, V) the corresponding automorphic representation with the automorphic functional $I \in V^*$ and the Helgason-Zelditch distribution $e_\pi \in \mathcal{D}(X)$ (see (12)). Let also $a(g) = a(z,b) \in C^\infty(\Gamma \setminus D \times B)$ be a symbol of order zero (which we assume is independent of λ).

In [Z2], on the basis of the representation (8), Zelditch defined the corresponding pseudo-differential operator $A = Op(a) : C^\infty(Y) \to C^\infty(Y)$ by

$$Op(a)\phi(z) = \int_B a(z,b) e^{(\frac{1+\lambda}{2})<z,b>} dT(b) \ . \tag{22}$$

We can rewrite this in the form

$$Op(a)\phi(z) = \int_K a(gk) e_\pi(gk) dk \ . \tag{23}$$

Hence, the action of A on ϕ reduces to the multiplication of the corresponding distribution e_π by the symbol $a(g)$ and then taking the K-invariant part of the result.

5.2. **Distributions** dU_i**.** Interpreting pseudo-differential operator as an observable in Quantum Mechanics one is led to the introduction of correlation functions or matrix coefficients. Namely, one is interested in studying following quantities

$$A_{ij} =< Op(a)\phi_i, \phi_j > \ . \tag{24}$$

One view these as distributions on the space of symbols. We will concentrate on the diagonal terms $< Op(a)\phi_i, \phi_i >$ first. This leads us to the following definition of distributions dU_i on X associated to eigenfunctions ϕ_i on Y:

$$< Op(a)\phi_i, \phi_i >:= \int_X a(x) dU_i \ . \tag{25}$$

Using the interpretation (23) we arrive to the following defining relation for the distributions dU_i:

$$\int_X a(x) e_{\pi_i}(x)\bar{\phi}_i(x) dx := \int_X a(x) dU_i \ . \tag{26}$$

Hence we see that

$$dU_i = e_{\pi_i}(x)\bar{\phi}_i(x) \tag{27}$$

as distributions on X. Note that from the construction of e_{π_i} it follows that $\int_X 1 dU_i = \int_X |\phi_i|^2 dx = 1$.

5.3. **Automorphic functions as symbols.** We now rephrase Theorem 1.2 from the Introduction in terms of automorphic representations (while in the Introduction we stated it in equivalent terms of eigenspaces of the Casimir; see Section 3). This theorem underlies our study of action of pseudo-differential operators on eigenfunctions.

Theorem. *Let $\nu_\mu : C^\infty(S^1) \to V_\mu \subset C^\infty(X)$ be an irreducible automorphic representation and the corresponding G-morphism. For any $a \in V_\mu$ and any Maass forms ϕ_i and ϕ_j there are a constant a_{μ,μ_i,μ_j} and an explicit distribution $l_{\mu,\mu_i,\mu_j} \in \mathcal{D}(S^1)$, depending only on μ, μ_i, μ_j but not on the choice of the corresponding functions, such that the following relation holds*

$$< Op(a)\phi_i, \phi_j >_Y = a_{\mu,\mu_i,\mu_j} \cdot < l_{\mu,\mu_i,\mu_j}, v_a >_{V_\mu}, \tag{28}$$

where $a = \nu_\mu(v_a)$ with $v_a \in C^\infty(S^1)$.

Proof. We deal with symbols coming from the class one representations V_μ. The case of symbols coming from discrete series is similar and is explained in detail in the course of the proof of Theorem 6.2.

As the symbol $a \in V_\mu$ belongs to an irreducible representation we have from (23) and (15)

$$< Op(a)\phi_i, \phi_j >= \int_X \psi(x) e_{\pi_i}(x) \bar{\phi}_j(x) dx = l^{aut}_{\pi_\mu,\pi_i,\pi_j}(a, e_{\pi_i}, \phi_j) \ . \tag{29}$$

Hence from (13) and (20) we have

$$l^{aut}_{\pi_\mu,\pi_i,\pi_j}(a, e_{\pi_i}, \phi_j) = \ a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}} \cdot l^{mod}_{\pi_\mu,\pi_i,\pi_j}(v_a, \delta, e_0) \tag{30}$$

and hence $l^{mod}_{\pi_\mu,\pi_i,\pi_j}(v_a, \delta, e_0) =< l_{\mu,\mu_i,\mu_j}, v_a >_{V_\mu}$ could be viewed as the evaluation of the distribution with the explicit kernel on S^1, given by (38), on the function v_a once we consider the identification $C^\infty(S^1)_{even} \simeq V_\mu$.

$\square$

6. Invariance of dU_i under the geodesic flow

6.1. **Geodesic flow.** It is well known that under the identification $S^*(Y) \simeq X$ the geodesic flow G_t on $S^*(Y)$ corresponds to the right action on X of the diagonal subgroup $T = \{g_t = diag(e^t, e^{-t}) |\ t \in \mathbb{R}\} \subset G$ (see [GF]).

6.2. **Asymptotic invariance.** In order to prove asymptotic invariance of distributions dU_i we will show that for any symbol $a(x) \in C^\infty(X)$

$$\left| \int_X (a(x) - a(xg_t))\, dU_i \right| = O_{a,t}(|\lambda_i|^{-\alpha}) \ , \tag{31}$$

for some $\alpha > 0$ and with a uniform constant in the O-term as t changing in a compact set and a bounded (w.r.t. a Sobolev norm on $C^\infty(X)$). We will show that one can choose $\alpha = 1$ above. Such bounds are usually obtained as a consequence of the Egorov-type theorem (see [Z1]). Zelditch found another way to prove such bounds based on the exact differential relation satisfied by dU_i. We will use trilinear invariant functionals introduced above in order to prove (31). As a consequence of our proof we will be able to speculate (on the basis of Conjecture 4.4.2) that the true order of decay in (31) should be $|\lambda_i|^{-3/2}$. We note that the Egorov's theorem gives only $|\lambda_i|^{-1}$ as the order of decay in (31).

In order to be able to connect distributions dU_i to trilinear invariant functionals we will consider symbols which are themselves automorphic functions (i.e. symbols which belong to one of automorphic representations V_i). Such functions are dense in $C(X)$ (e.g. union of basis $\{\psi^i_k(x)\}$ of all spaces V_i) and hence describe dU_i uniquely.

We have the following

Theorem. *For any fixed automorphic representation (π_μ, V_μ) there exists an explicit constant c_μ such that for a given automorphic function (which we view as a symbol) $\psi(x) \in V_\mu \subset C^\infty(X)$ the following relation holds*

$$< Op(\psi)\phi_i, \phi_i >= a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}} |\lambda_i|^{-\frac{1}{2}} c_\mu d_\mu(\psi) \ + \ O_{\psi,\mu}(|\lambda_i|^{-1}) \ , \tag{32}$$

where d_μ is the properly normalized, independent of λ_i, T-invariant functional on V_μ. The constant in the O-term above is effective in μ and (the Sobolev norm of) ψ.

Corollary. *For ψ as above we have*

$$\left|\int_X (\psi(x)-\psi(xg_t))\,dU_i\right| = O_{\psi,t}(|\lambda_i|^{-1}) \ , \tag{33}$$

as $|\lambda_i|\to\infty$.

Remark. 1. From the proof it follows that the constant in the O-terms above is bounded by the second L^2-Sobolev norm of ψ and $|\mu|^{\frac{1}{2}}$. Hence we have for a general symbol $a\in C^\infty(X)$:

$$\left|\int_X (a(x)-a(xg_t))\,dU_i\right| \leq C_t\cdot S_2(a)\cdot|\lambda_i|^{-1} \ , \tag{34}$$

as $|\lambda_i|\to\infty$. Here S_2 is the second L_2-Sobolev norm on X. This should be viewed as a quantitative version of the asymptotic invariance of the distributions dU_i.

2. We have seen in (21) that $|a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}|\leq C|\lambda_i|^{\frac{1}{2}}$ and hence the coefficients in front of the distribution d_μ are uniformly bounded in λ_i. One expects that coefficients $a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}$ grow at a much slower rate (e.g. Conjecture 4.4.2). This is known as the (effective) Quantum Unique Ergodicity conjecture of Rudnick and Sarnak solution to which (in an ineffective form) was recently announced by E. Lindenstrauss [Li2].

6.3. **Proof.** In order to prove (32) we use our interpretation of pseudo-differential operators with automorphic symbols in terms of trilinear functionals. We first deal with symbols ψ coming from automorphic representations of class one described in 3.3. We have from (25) and (27)

$$< Op(\psi)\phi_i,\phi_i >= \int_X \psi(x)e_{\pi_i}(x)\bar{\phi}_x dx = l^{aut}_{\pi_\mu,\pi_i,\pi_i}(\psi,e_{\pi_i},\phi_i) \ . \tag{35}$$

Let $\psi=\nu_\mu(v)$ for $v\in V_\mu\simeq C^\infty_{even}$, $\delta\in V^*_{\lambda_i}$ the distribution which corresponds to e_{π_i} and $e_0=e_{0,\lambda_i}$ the K-fixed vector in V_{λ_i}. From the uniqueness of trilinear functionals (20) we arrive at

$$< Op(a)\phi_i,\phi_i >= a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}\cdot l^{mod}_{\pi_\mu,\pi_i,\pi_i}(v,\delta,e_0) \ . \tag{36}$$

We use now the explicit description of l^{mod} in (18) in order to compute the right hand part of (36). We have (recall that e_0 is the constant function)

(37)

$$l^{mod}_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}(v,\delta,e_0) =$$
$$(2\pi)^{-3}\int_{(S^1)^3} v(x)\delta(y)\quad|\sin(y-z)|^{(\alpha-1)/2}|\sin(x-z)|^{(\beta-1)/2}|\sin(x-y)|^{(\gamma-1)/2}\ dxdydz,$$

where $x, y, z \in S^1$ are the standard angular parameters on the circle and $\alpha = -2\lambda_i + \mu,\ \beta = -\mu,\ \gamma = -\mu$ as before .

We note that $\delta = \delta_0$ is the Dirac delta at $y = 0$ and hence we need to compute the following integral

$$\int_{(S^1)^2} v(x)|\sin(z)|^{-\lambda_i + \frac{1}{2}\mu - \frac{1}{2}}|\sin(x-z)|^{-\frac{1}{2}\mu - \frac{1}{2}}|\sin(x)|^{-\frac{1}{2}\mu - \frac{1}{2}}\ dxdz\ . \tag{38}$$

as $|\lambda| = |\lambda_i| \to \infty$. We compute it with the help of the stationary phase method. We write (38) as

$$\int_{(S^1)^2} A(x,z)e^{i\lambda p(z)}dxdz\ , \tag{39}$$

with the amplitude

$$A(x,z) = v(x)|\sin(z)|^{\frac{1}{2}\mu - \frac{1}{2}}|\sin(x-z)|^{-\frac{1}{2}\mu - \frac{1}{2}}|\sin(x)|^{-\frac{1}{2}\mu - \frac{1}{2}} \tag{40}$$

and the phase (which depends only on z)

$$p(z) = -\ln|\sin(z)|\ . \tag{41}$$

A direct computation shows that the phase p has two non-degenerate critical points $z_\pm = \pm\pi/2$ with the equal contribution to the integral because of the symmetry (all functions involved are even on S^1). We note now that this explains why the integral (or more precisely its leading term) (37) gives rise to a distribution on $S^1 \times S^1$ which is invariant under the diagonal action of the diagonal subgroup T. The simple geometric reason for this is that by the stationary phase method the leading term is given by the value at the critical points $y = 0,\ z = \pi/2$ and $y = 0,\ z = -\pi/2$. These points are fixed points of the diagonal action of T. Near these fixed points elements of T contract (or expand) in direction of y and expand (respectively contract) by the same amount in direction of z. Hence, distribution supported in these fixed points is invariant with respect to the diagonal action in the space $V_{\lambda_i} \times V_{\lambda_i} \simeq C^\infty(S^1 \times S^1)$.

On the formal level, the second derivative of the phase at the fixed point is 1 and the value of the phase is 0 and hence the (equal) contribution from critical points to the integral (39) is given by $|\lambda|^{-\frac{1}{2}}$ times the value of

$$\int_{S^1} v(x)|\sin(x)|^{-\frac{1}{2}\mu - \frac{1}{2}}|\sin(x - \pi/2)|^{-\frac{1}{2}\mu - \frac{1}{2}}dx = \int_{S^1} v(x)|\sin(2x)|^{-\frac{1}{2}\mu - \frac{1}{2}}dx\ , \tag{42}$$

which is exactly a T-invariant distribution on V_μ. Let d_μ be the unique T-invariant distribution on V_μ normalized to have value one on the K-fixed vector e_0. The value of the functional in (42) on e_0 (i.e. on the constant function 1) is given by the classical integral and gives the main term in (32)

$$c_\mu = \int_{S^1} |\sin(2x)|^{-\frac{1}{2}\mu - \frac{1}{2}}dx\ =\ \frac{2^{-\frac{1}{2} + \frac{1}{2}\mu}\Gamma(\frac{1}{2} - \frac{1}{2}\mu)}{\Gamma(\frac{3}{4} - \frac{1}{4}\mu)^2}\ . \tag{43}$$

We note that all but finite number of eigenvalues of Δ are greater than $\frac{1}{4}$ and hence all but finite number of μ's are purely imaginary. From the Stirling's formula we have $|c_\mu| = O(|\mu|^{\frac{1}{2}})$ as $|\mu| \to \infty$.

The reminder in the stationary phase method is of order $O_v(|\lambda|^{-3/2})$. Here the constant in the O-term is bounded by the first derivative of v. From (21) we see that $|a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}| \ll |\lambda_i|^{\frac{1}{2}}$ and hence the O-term claimed.

We now turn to symbols coming from the discrete series representations. Let $k > 1$ be an odd positive integer and D_k the space (of smooth vectors) of the corresponding discrete series representation (see [G5] and [L] for various descriptions of discrete series). We will use the following well-known realization of discrete series. Let V_{-k} be the space of smooth functions of the homogeneous degree $-k-1$ on $\mathbb{R}^2$. The space D_k could be realized as a subspace in V_{-k} (with the quotient isomorphic to the finite-dimensional representation of the dimension k; see [G5]). We denote the corresponding imbedding by $i_k : D_k \to V_{-k}$.

For a representation V_{λ_i} of the principal series we need to construct a (unique) G-invariant functional on the tensor product $D_k \otimes V_{\lambda_i} \otimes V_{\lambda_i}$. We first note that the formula (16) defines the kernel of the G-invariant functional on the (reducible) representation $V_{-k} \otimes V_{\lambda_i} \otimes V_{\lambda_i}$. For this one can use general methods of analytic continuation of integrals described in [G1] to regularize the integral in (18). This gives meaning to the value of this integral for $\mu = -k$. This is true for any λ which is not a pole of the analytic continuation of (18). For a value of λ which is a pole of (18) one also can assign an invariant functional by taking the residue. It is easy to see that $\mu = -k$ is not a pole for the analytic continuation. This is especially easy to see for λ non-real. Hence we have an invariant functional on $V_{-k} \otimes V_{\lambda_i} \otimes V_{\lambda_i}$ which gives rise to the invariant functional on the subspace $D_k \otimes V_{\lambda_i} \otimes V_{\lambda_i}$. It is easy to see that such a functional is non-zero on D_k for $k \equiv 3 \pmod 4$. For $k \equiv 1 \pmod 4$ this functional vanishes on D_k and one have to consider the derivative of $l_{\mu,\lambda_i,\lambda_i}$ in μ evaluated at $\mu = -k$. In both cases the functionals obtained are very similar. We consider the case $k \equiv 3 \pmod 4$ for simplicity and leave the similar case $k \equiv 1 \pmod 4$ to the reader. The value of the described above invariant functional on the triple $v \in D_k$, $\delta,\ e_0 \in V_{\lambda_i}$ is defined by the analytic continuation of the integral

$$\int_{(S^1)^2} v(x)|\sin(z)|^{-\lambda_i-\frac{1}{2}k-\frac{1}{2}}|\sin(x-z)|^{\frac{1}{2}k-\frac{1}{2}}|\sin(x)|^{\frac{1}{2}k-\frac{1}{2}}\, dxdz\ . \tag{44}$$

The value of (44) is obtained by the analytic continuation of the distribution f_s on S^1 which is given by the kernel $|\sin(z)|^s$ for $Re(s) > -1$ and then analytically continued to $s = -\lambda_i - \frac{1}{2}k - \frac{1}{2}$. Moreover, the contribution from a small neighborhood of singular points $(z = 0,\ \pi)$ to the value of f_s on any fixed smooth function is negligible as $Im(s) \to \infty$. Namely, let $g(z) \in C^\infty(S^1)$ be a smooth function with a support in $(-0.1\pi, 0.1\pi)$ then $|f_s(g)| \ll |Im(s)|^{-N}$ for any $N > 0$ as $Im(s) \to \infty$. This implies that we can disregard any small enough, fixed neighborhood of $z = 0$ in the integral (44) and hence we end up with the integral without non-integrable singularities. Such an integral could be treated in the same fashion as before and hence the leading term is given by $|\lambda_i|^{-\frac{1}{2}}$ times the

value of the integral

$$\int_{S^1} v(x)|\sin(2x)|^{\frac{1}{2}k-\frac{1}{2}}dx\ . \tag{45}$$

This is again the unique (up to a constant) T-invariant distribution on D_k. Taking into account that the reminder in the stationary phase method is of order $|\lambda|^{-3/2}$ we arrive to the reminder in (33) for a fixed k. We study now the dependence on k of the constant in the reminder. For this we normalize this T-invariant distribution by computing its value on a special vector in D_k. Namely, let $w_k = \exp(i(-1-k))$ be the highest weight vector in D_k (strictly speaking w.r.t. $PSL_2(\mathbb{R})$). Let d_k be the distribution taking the value 1 on a unit vector proportional to w_k. The value of (45) on w_k is given by the classical integral (due to Ramanujan, see [Ma])

$$\int_{S^1} |\sin(2x)|^{\frac{1}{2}k-\frac{1}{2}}e^{i(-1-k)x}dx \ = \ e^{-\frac{1}{4}i\pi(k+1)}\frac{2^{\frac{1}{2}-\frac{1}{2}k}\Gamma(\frac{1}{2}+\frac{1}{2}k)}{\Gamma(1+\frac{1}{2}k)\Gamma(\frac{1}{2})}\ . \tag{46}$$

From this we see that the last expression is of order $\alpha_k = \pi^{-\frac{1}{2}}2^{\frac{1}{2}-\frac{1}{2}k}|k|^{-\frac{1}{2}}$. Taking into account that $||w_k||^2_{D_k} = \Gamma(2k)^{-1}$ (see [G5]) we arrive at $|c_k| = \pi^{-\frac{1}{2}}2^{\frac{1}{2}-\frac{1}{2}k}|k|^{-\frac{1}{2}}\Gamma(2k)^{\frac{1}{2}}$.

To estimate the reminder we need to estimate the automorphic coefficient $a_{\pi_k,\pi_{\lambda_i},\pi_{\lambda_i}}$. We will show that the bound

$$|a_{\pi_k,\pi_{\lambda_i},\pi_{\lambda_i}}|^2 = O(|k|2^k\Gamma(2k)^{-1}) \tag{47}$$

holds. This bound is similar to the bound (21). The appearance of the Γ-function is due to the awkward normalization of the trilinear functional for the discrete series. This is mostly due to the author's lack of knowledge of good models of discrete series. We expect that a stronger bound follows from methods of [BR3]. We show here how to obtain the claimed bound by a more elementary means.

Let $k \ll |\lambda|$. We estimate the value of the model trilinear invariant functional $l^{mod}_{\pi_k,\pi_\lambda,\pi_\lambda}$ on specially chosen (smooth) vectors. For the automorphic trilinear functional we use the bound coming from the maximum modulus estimate on vectors in the automorphic representation D_k. This will give us a bound on the coefficient of proportionality $a_{\pi_k,\pi_{\lambda_i},\pi_{\lambda_i}}$.

We choose the triple $w_k\otimes e_0\otimes e_{k+1}$, where w_k is as above and $e_l \in V_{\lambda_i}$ is a unit vector of the K-type l which we will view as a function $e_l = \exp(il\theta)$ in the realization $V_{\lambda_i} \simeq C^\infty(S^1)$. We note that since this triple is invariant under the action of the diagonal copy of K the integral we have to compute could be reduced to

$$\int_{(S^1)^2} e_0(y)e_{k+1}(z)|\sin(y-z)|^{-\lambda_i-\frac{1}{2}k-\frac{1}{2}}|\sin(y)|^{\frac{1}{2}k-\frac{1}{2}}|\sin(z)|^{\frac{1}{2}k-\frac{1}{2}}\ dydz\ . \tag{48}$$

As before the stationary phase method imply that the main contribution to this integral is given by $|\lambda|^{-\frac{1}{2}}$ times the value of the integral along the line $x-y=\pi/2$, namely

$$\int_{S^1} |\sin(2x)|^{\frac{1}{2}k-\frac{1}{2}}e^{i(k+1)x}dx \tag{49}$$

which we computed above and saw that it is of order of $\alpha_k = \pi^{-\frac{1}{2}}2^{\frac{1}{2}-\frac{1}{2}k}|k|^{-\frac{1}{2}}$. As we mentioned the norm of w_k is equal to $\Gamma(2k)^{-\frac{1}{2}}$ and hence using Sobolev type bound from [BR2] we arrive at the pointwise bound for the automorphic realization $\phi_k(g) = \nu_k(w_k)(g)$ of the highest weight vector w_k in the discrete series D_k of the type $\sup_X |\phi_k| \leq C|k|^{\frac{1}{2}}\Gamma(2k)^{-\frac{1}{2}}$ and hence the bound on $a_{\pi_k,\pi_{\lambda_i},\pi_{\lambda_i}}$ claimed in (47). Combined with the computed value for c_k, this gives the bound for the constant in the reminder in (33).

$\square$

7. Non-negative microlocal lifts

We now want to correct distributions dU_i by a smaller order term in λ_i (as $|\lambda_i| \to \infty$) in order to obtain probability measures dm_i on X. Namely, we want to construct a family of probability measures dm_i such that for any $f \in C^\infty(X)$ the following relation holds

$$\int_X f(x)dU_i = \int_X f(x)dm_i \ + \ O_f(|\lambda_i|^{-\frac{1}{2}}) \ . \tag{50}$$

This is usually done by means of averaging over a small set in the phase space. Such a procedure is called Friedrichs symmetrization ([Sh],[CdV],[Z2]). However, Friedrichs symmetrization does not commute with the action of G and hence does not preserve automorphic representations (this problem is discussed in [Z3]). In this section we show that one can exhibit a variety of families of probability measures which are asymptotic to dU_i and constructed via representation theory.

7.1. **Probability measures.**

We construct asymptotic to dU_i probability measures on X by taking restrictions of automorphic functions $\psi\otimes\bar{\psi} \in V_{\lambda_i}\otimes V_{\lambda_i}$ on $X\times X$ to the diagonal $\Delta X \hookrightarrow X\times X$. Where $\psi \in V_{\lambda_i}$ is a specially chosen L^2-normalized automorphic functions. This will give rise to a probability measures since representations V_{λ_i} are self-dual and hence the resulting function is non-negative on ΔX. Our construction is motivated by Wolpert's approach to the microlocal lift via the Fejér kernel ([Wo]).

Let $\chi(t) \in C^\infty(S^1)$ be a smooth non-negative function supported in $(-\pi/4, \pi/4)$ and with the norm $\int |\chi(t)|^2 dt = 1$. We consider a family of vectors $v_r \in V_{\lambda_i} \simeq C^\infty_{even}(S^1) \simeq C^\infty(S^1)$ for $r > 1$ defined by $v_r = 2^{-\frac{1}{2}}r^{\frac{1}{2}}(\chi(rt) + \chi(r(t-\pi/2)))$, $t \in S^1$ (i.e. the sum of two contracted bump functions around 0 and around $\pi/2$). We note that $||v_r|| = 1$. Clearly the function $\rho_r(x) = \rho_r^{\lambda_i}(x) = \nu(v_r)\otimes\nu(v_r)\ |_{\Delta X}$ is a density of a probability measure on X. We have the following

Theorem. *For any (symbol) $\psi \in C^\infty(X)$ and any $\varepsilon > 0$ there exists an effective constant $C = C_{\psi,\varepsilon}$ such that for any λ_i we have*

$$\left|\int_X \psi(x)dU_i \ - \ \int_X \psi(x)\rho_r(x)dx\right| \ \leq C|\lambda_i|^{-\frac{1}{2}} \ , \tag{51}$$

as $r \to \infty$ and $|\lambda_i| \to \infty$ condition to $r \leq |\lambda_i|^{\frac{1}{2}-\varepsilon}$.

Proof. Consider a given $r > 1$. We may assume that $\psi \in V_\mu$ and $\psi = \nu(v)$ as in 3.1. The integration in (32) is over the small neighborhood (depending on the value of r) of four points $x = 0$ or $\frac{1}{2}\pi$ and $y = 0$ or $\frac{1}{2}\pi$. However, as we saw in the proof of Theorem 6.2, only two points $(x, y) = (0, \frac{1}{2}\pi)$ and $(\frac{1}{2}\pi, 0)$ are the stationary points of the phase and hence only these contribute to the leading term of (32). Moreover this contribution was computed in the course of the proof of Theorem 6.2. This contribution is coming from an integral of v over the neighborhood of the size smaller than $|\lambda_i|^{-\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$. This again follows from the stationary phase method. The function v is well approximated on a small enough interval by its value in the center of this interval. Hence by letting $r \to \infty$ but keeping it smaller than $|\lambda_i|^{\frac{1}{2}-\varepsilon'}$ with $\varepsilon' > \varepsilon$ we see that for any smooth function ψ the value of the integral against ρ_r has the same leading term as the integral against dU_i as $|\lambda_i| \to \infty$ and $r \to \infty$. The constant $C_{\psi,\varepsilon}$ in the O-term is bounded by appropriate derivative of v at stationary points.

□

8. Spectral localization of eigenfunctions under the action of PDO

8.1. **Spectral localization.** We consider now more general matrix coefficients. Let $a \in C^\infty(X)$ be a symbol and $Op(a)$ the corresponding pseudo-differential operator. We assume for simplicity that a belongs to an automorphic representation of class one. For a fixed symbol a we are interested in the decomposition of $Op(a)\phi_i$ with respect to the basis of eigenfunctions $\{\phi_j\}$ as $|\lambda_i| \to \infty$.

Theorem. *For for a fixed symbol $a \in C^\infty(X)$ and for any $N > 0$ the following bound holds*

$$|< Op(a)\phi_i, \phi_j >| = O_N(|\lambda_i - \lambda_j|^{-N}) \tag{52}$$

with the constant in the O-term depends on N and on the symbol.

Proof. To prove (52) we need to analyze the values of

$$l^{aut}_{\pi_\mu,\pi_i,\pi_j}(a, e_{\pi_i}, \phi_j) = a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_j}} l^{mod}_{\pi_\mu,\pi_i,\pi_j}(a, e_{\pi_i}, \phi_j)\ . \tag{53}$$

We saw that coefficients $a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_j}}$ are polynomially bounded in λ_i. On the other hand it follows from the stationary phase method that the structure of the model trilinear invariant functionals l^{mod} is governed by the presence of critical points of the phase and singularities of the amplitude. It is easy to see that as $|\lambda_i - \lambda_j| \to \infty$ the phase function in the kernel of l^{mod} does not have critical points with respect to x-integration in (37) and non-degenerate critical point with respect to z-integration. The amplitude of the kernel is becoming a smooth function after integration against the smooth function a. Hence from the stationary phase method and the bound (21) on coefficients $a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_j}}$ we obtain the bound claimed. □

8.2. **Conjectural density.** According to the proposition above we see that for a fixed symbol a the spectral density of $Op(a)\phi_i$ is essentially supported on a very short interval around λ_i itself. Hence, under the action of $Op(a)$ the eigenfunctions ϕ_i are spectrally localized in short intervals.

This makes a question about spectral density of $Op(a)\phi_i$ inside the interval $|\lambda_i - \lambda_j| \ll |\lambda_i|^\varepsilon$ interesting. We note that this question was also raised in a connection with the quantum unique ergodicity conjecture. We may speculate about the size of coefficients $< Op(a)\phi_i, \phi_j >$ on the basis of a conjecture similar to Conjecture 4.4.2. One is lead to conjecture (though, solely on the basis of examples of arithmetic surfaces, see [Sa]) that the coefficients $a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_j}}$ are of the order $\max\{|\lambda_i|^\varepsilon, |\lambda_j|^\varepsilon\}$ for any $\varepsilon > 0$. On the other hand it is also expected that these coefficients are not small on the average (though some of them could be zero). Namely, one expects, for example, that for all λ_i and any fixed $B > 0$ the lower bound

$$\sum_{|\lambda_i - \lambda_j| \leq B} |a_{\pi_\mu,\pi_{\lambda_i},\pi_{\lambda_i}}|^2 \geq c|\lambda_i| \ , \tag{54}$$

holds for some $c > 0$. This again is consistent with the Lindelöf conjecture since according to the Weyl law the number of terms in the sum above is of order $|\lambda_i|$.

On the other hand it is easy to see from the stationary phase method that

$$|l^{mod}_{\pi_\mu,\pi_i,\pi_j}(a, e_{\pi_i}, \phi_j)| \asymp |\lambda_i|^{-\frac{1}{2}} \tag{55}$$

for $|\lambda_i - \lambda_j| \leq B$. We deduce from this that for a fixed symbol a the spectral density of $Op(a)\phi_i$ is supported in the interval $|\lambda_i - \lambda| \ll |\lambda_i|^\varepsilon$ (as we have shown in Theorem 8.1) and conjecture that it has the absolute value of order $|\lambda_i|^{-\frac{1}{2}+\varepsilon}$ at most on this interval. Hence, $Op(a)$ spreads ϕ_i evenly on this interval.

Similarly, the conjectural upper bound for the coefficients $a_{\pi_\mu,\pi_{\lambda_i}\pi_{\lambda_j}}$ implies that the matrix coefficients satisfy

$$| < Op(a)\phi_i, \phi_j > | \ll |\lambda_i|^{-\frac{1}{2}} \ , \tag{56}$$

as $|\lambda_i - \lambda_j| \to 0$ and $|\lambda_i| \to \infty$.

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Bar-Ilan University, Ramat Gan, Israel

E-mail address: reznikov@math.biu.ac.il